MONDAY, MAY 3rd. PRESS RECORD

(Next Class: Quiz on §2.3–2.5)

LAST CLASS...

Gradients & Directional Derivatives.

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ (or $\mathbb{R}^3 \longrightarrow \mathbb{R}$).

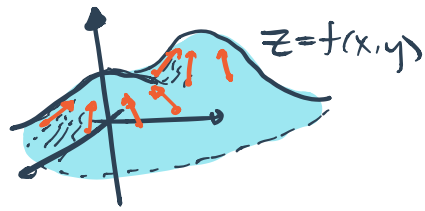
The directional derivative of f at a point $(x_0, y_0) \in \mathbb{R}^2$ in the direction of a vector $\bar{v}$ is the number

$$\nabla f(x_0, y_0) \cdot \bar{v}$$

MOTTO. $\nabla f(x_0, y_0) \cdot \bar{v}$ is the rate of change of f at the point (x_0, y_0) if you move in the $\bar{v}$-direction.

Theorem. The gradient ∇f points in the direction that f is increasing the fastest.

GRAPHICALLY. If you graph



The gradient ∇f points in the steepest ascending direction.

Exercise. Let $f(x,y) = x + e^{x-y}$.

① In what direction does f increase most quickly at $(x,y) = (0,0)$?

② COMPUTE the directional derivative at $(0,0)$ in the direction $(1,2)$.

ANS #1. The direction that f is increasing most quickly is given by

$$\nabla f(0,0) = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\right)$$

$$\frac{\partial f}{\partial x} = 1 + e^{x-y} \qquad \frac{\partial f}{\partial y} = -e^{x-y}$$

$$\nabla f(0,0) = (2,-1).$$

ANS #2 The directional derivative is:

$$\nabla f(0,0) \cdot (1,2) = (2,-1) \cdot (1,2)$$

$$= 2 - 2 = 0.$$

THEOREM. Let $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a function. Assume $f(x_0, y_0, z_0) = c$ (so (x_0, y_0, z_0) lies on the level set of level c), if

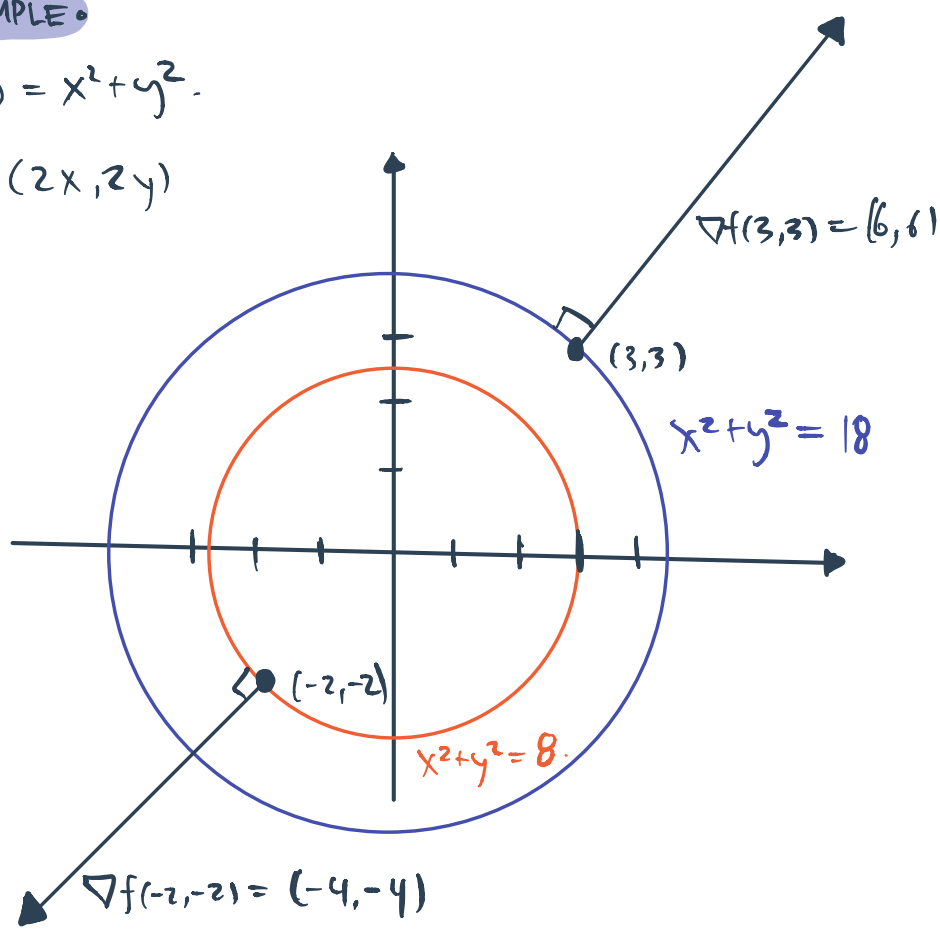
$$\nabla f(x_0, y_0, z_0) \neq (0,0,0)$$

then it is a normal vector to the level set of level c.

EXAMPLE.

$f(x,y) = x^2 + y^2$.

$\nabla f = (2x, 2y)$



TANGENT LINES / PLANES TO LEVEL SETS...

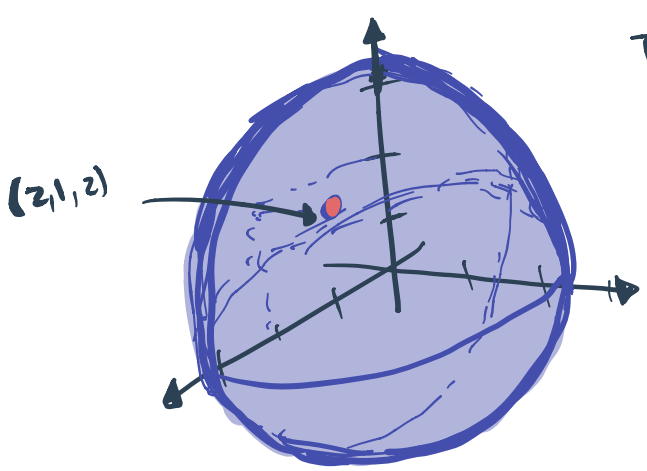
If $f(x_0, y_0, z_0) = c$ and $\nabla f(x_0, y_0, z_0) \neq (0,0,0)$ then it gives a normal vector to the level set...

equation for tangent plane:

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

EXAMPLE. $x^2 + 2y^2 + z^2 = 10$ is the level 10 level set of $f(x,y,z) = x^2 + 2y^2 + z^2$.

The point $(2,1,2)$ is on this level set:



Tangent plane at $(2,1,2)$

$$\nabla f(2,1,2) \cdot (x-2, y-1, z-2) = 0.$$

$$\nabla f = (2x, 4y, 2z)$$

$$\nabla f(2,1,2) = (4,4,4)$$

eqn for tangent plane: $(4,4,4) \cdot (x-2, y-1, z-2) = 0.$

$$4(x-2) + 4(y-1) + 4(z-2) = 0.$$

PLOTTING 2D GRADIENTS

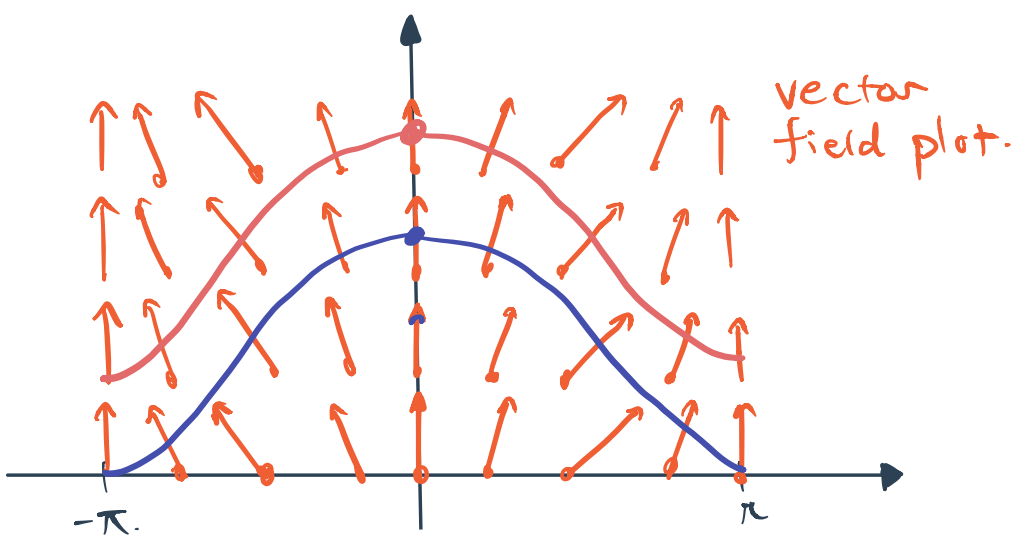
$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

The gradient assigns each point in $\mathbb{R}^2$ a vector.

EXAMPLE.

$$f(x,y) = y - \cos(x).$$

$$\nabla f = (\sin(x), 1).$$



level 1:

$y - \cos(x) = 1.$

$y = 1 + \cos(x)$

level 2:

$y - \cos(x) = 2$

$y = \cos(x) +$

§3.1 • Iterating Partial Derivatives

If $f(x,y)$ is twice differentiable (or a C^2-function) then there are 4 2nd derivatives we can take:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$$

We put these in a 2×2 matrix, called the Hessian of f:

$$\mathrm{Hess}(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

THEOREM. If f is C^2, then:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

So... order of differentiation doesn't matter for nice functions.

Exercise. Let $f(x,y) = x^3 + xy$.
COMPUTE $\mathrm{Hess}(f)$ at the point $(1,1)$.

ANS. $\frac{\partial^2 f}{\partial x^2} = 6x$ $\quad \frac{\partial^2 f}{\partial x \partial y} = 1.$

$$\frac{\partial^2 f}{\partial y^2} = 0.$$

$$\mathrm{Hess}(f) = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathrm{Hess}(f)(1,1) = \begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}.$$